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Special case of Asymptotic Eigenvalues of Second order Differential Equations with Three Turning Points and Neumann conditions

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ABSTRACT: In this present paper I concerned the equation $-W'' + q(x)W = \lambda\phi^2(x)W,$ $\phi^2(x) = \prod_{i=1}^{3}(x - x_i)^{l_i}\phi_0(x), 0 \prec x_1 \prec x_2 \prec x_3 \prec 1,$ with three turning points and Neumann conditions $W'(0) = W'(1) = 0.$ I have obtained the asymptotic eigenvalues when $x \in (x_1 \ , \ x_2).$.

Keywords: Asymptotic eigenvalues, Turning points, Neumann conditions.

INTRODUCTION

Let consider the boundary value problem of the form

$$-W'' + q(x)W = \lambda\phi^2(x)W \qquad for \qquad x \in I = [0\ ,1] \qquad (1)$$

where $\lambda = \rho^2$ is the spectral parameter; $\phi^2(x)$ and $q(x)$ are real functions. We suppose that

$$\phi^2(x) = \prod_{i=1}^{3}(x - x_i)^{l_i}\phi_0(x) \qquad (2)$$

where, $0 < x_1 < x_2 < x_3 < 1,\ l_i \in N,\ \phi_0(x) > 0$ for $x \in I = [0\ ,1]$ and $\phi_0(x)$ is twice continuously differentiable function on I. On other words, $\phi^2(x)$ has in three zeros $x_i\ , i = 1\ ,2\ ,3$ of order l_i, the zeros x_i of $\phi^2(x)$ are called turning points. In this section we obtained the asymptotic eigenvalues of equation (1) in three turning points with Neumann conditions $W'(0) = W'(1) = 0$.

Notations

In the real second-order differential equation

$$-W'' + q(x)W = \lambda\phi^2(x)W \qquad for \qquad x \in I = [0\ ,1] \qquad (3)$$

$\phi^2(x)$ has in I, there zeros x_i of order $l_i\ ,\ i = 1\ ,2\ ,3$ where, l_1 is even, l_2 is odd and l_3 is even. Let $\varepsilon > 0$ be fixed sufficiently small and let

$$D_{i,\varepsilon}=[x_i+\varepsilon\ ,\ x_{i+1}-\varepsilon]\ ,\quad i=1\ ,\ 2\qquad D_{3,\varepsilon}=[x_3+\varepsilon\ ,\ 1] \tag{4}$$
$$I_{i,\varepsilon}=[x_{i-1}+\varepsilon\ ,\ x_i-\varepsilon]\cup[x_i-\varepsilon\ ,\ x_{i+\varepsilon}]\cup[x_{i+\varepsilon}\ ,\ x_{i+1}-\varepsilon].$$

In [1] distinguished four different type of turning points: for $1\le\nu\le m$

$$T_\nu=\begin{cases} I & if\ \ l_\nu\ \ is\ even\ and\quad \phi^2(x)(x-x_\nu)^{-l_\nu}<0\quad in\ I_{\nu\varepsilon},\\ II & if\ \ l_\nu\ \ is\ even\ and\quad \phi^2(x)(x-x_\nu)^{-l_\nu}>0\quad in\ I_{\nu\varepsilon},\\ III & if\ \ l_\nu\ \ is\ odd\ and\quad \phi^2(x)(x-x_\nu)^{l_\nu}<0\quad in\ I_{\nu\varepsilon},\\ IV & if\ \ l_\nu\ \ is\ odd\ and\quad \phi^2(x)(x-x_\nu)^{l_\nu}>0\quad in\ I_{\nu\varepsilon}.\end{cases}$$

We know from [1] that x_1 is type of I , x_2 is type of IV and x_3 is of type II .

$$\mu_i=\frac{1}{2+l_i}\ ,\qquad \theta=\min\{\mu_1\ ,\ \mu_2\ ,\ \mu_3\}$$

The sectors S_{-1} is in form of

$$S_{-1}=\{\rho\,|\arg\rho\in[-\frac{\pi}{4}\ ,\ 0]\}.$$

We use for convenience the abbreviations

$$[1]=1+O(\frac{1}{\rho^\theta}).$$

The fundamental system solutions for $x\in I=[0\ ,\ 1]$

Now, let $W(x\ ,\ \lambda)$ be the solution of equation (1). The fundamental system of solutions (FSS) for equation (1) , when x_1 can be represented in the form (see [1] page 219)
since x_1 is type of I , we have the following FSS for $\rho\in S_{-1}$

$$W^{II}_{\nu_1}(x\ ,\ \rho)=\begin{cases}|\phi(x)|^{-\frac12}e^{i\rho\int_{x_\nu}^x|\phi(t)|dt}[1] & x\in D_{\nu-1,\varepsilon}\\ |\phi(x)|^{-\frac12}\csc\pi\,\mu_\nu\{e^{i\rho\int_{x_\nu}^x|\phi(t)|dt}[1]+i\cos\pi\,\mu_\nu e^{-i\rho\int_{x_\nu}^x|\phi(t)|dt}[1]\} & x\in D_{\nu,\varepsilon}\end{cases} \tag{5}$$

$$W^{II}_{\nu_2}(x\ ,\ \rho)=\begin{cases}|\phi(x)|^{-\frac12}e^{-i\rho\int_{x_\nu}^x|\phi(t)|dt}[1]+i\cos\pi\,\mu_\nu e^{i\rho\int_{x_\nu}^x|\phi(t)|dt}[1] & x\in D_{\nu-1,\varepsilon}\\ |\phi(x)|^{-\frac12}(\sin\pi\,\mu_\nu e^{-i\rho\int_{x_\nu}^x|\phi(t)|dt}[1]) & x\in D_{\nu,\varepsilon}\end{cases} \tag{6}$$

If x_ν be a turning point of type I , then the estimates for $W^I_{\nu_1}(x\ ,\ \rho)\ ,\ W^I_{\nu_2}(x\ ,\ \rho)$ are obtained from the corresponding estimates for $W^I_{\nu_1}(x\ ,\ \rho)\ ,\ W^I_{\nu_2}(x\ ,\ \rho)$ by substituting there in ρ by $i\rho$. The FSS of (1) for x_1 that is type of I whit sector S_{-1} are the following form

$$W_{1,1}(x\,,\,\rho)=\begin{cases}\left|\phi(x)\right|^{-\frac{1}{2}}e^{\rho\int_{x_1}^{x}|\phi(t)|dt}[1] & 0\le x<x_1\\ \left|\phi(x)\right|^{-\frac{1}{2}}\csc\pi\,\mu_1 e^{\rho\int_{x_1}^{x}|\phi(t)|dt}[1] & x_1<x<x_2\end{cases}\qquad(7)$$

$$W_{2,1}(x\,,\,\rho)=\begin{cases}\left|\phi(x)\right|^{-\frac{1}{2}}e^{-\rho\int_{x_1}^{x}|\phi(t)|dt}[1] & 0\le x<x_1\\ \left|\phi(x)\right|^{-\frac{1}{2}}\sin\pi\,\mu_1 e^{-\rho\int_{x_1}^{x}|\phi(t)|dt}[1] & x_1<x<x_2,\end{cases}\qquad(8)$$

The *Wronskian* of FSS satisfies in following form $W(\rho)=W(W_{1,1}(x\,,\,\rho)\,,\,W_{2,1}(x\,,\,\rho))=-2\rho[1].$

Asymptotic form of the solutions for $W'(0)=W'(1)=0$

Let us consider the differential equation (1) with following boundary conditions

$$C(0\,,\,\lambda)=1\,,\,C'(0\,,\,\lambda)=0.\qquad(9)$$

By applying the FSS $V_{1,1}(x\,,\,\rho)\,,\,V_{2,1}(x\,,\,\rho)$, for $x\in I_{1,\varepsilon}$ we have

$$C(x\,,\,\rho)=c_1W_{1,1}(x\,,\,\rho)+c_2W_{2,1}(x\,,\,\rho).$$

By derivation from $C(x\,,\,\rho)$ we can write

$$C'(x\,,\,\rho)=c_1W'_{1,1}(x\,,\,\rho)+c_2W'_{2,1}(x\,,\,\rho)\qquad for\quad x\in I_{1,\varepsilon}.$$

$$\begin{cases}C(x\,,\,\rho)=c_1W_{1,2}(x\,,\,\rho)+c_2W_{2,1}(x\,,\,\rho)\\ C'(x\,,\,\rho)=c_1W'_{1,2}(x\,,\,\rho)+c_2W'_{2,1}(x\,,\,\rho)\end{cases}\Rightarrow\begin{cases}C(0\,,\,\rho)=c_1W_{1,2}(0\,,\,\rho)+c_2W_{2,1}(0\,,\,\rho)=1\\ C'(0\,,\,\rho)=c_1W'_{1,2}(0\,,\,\rho)+c_2W'_{2,1}(0\,,\,\rho)=0\end{cases}\qquad(10)$$

We infer by using Cramer's rule leads to the following equation

$$C(x\,,\,\rho)=\frac{1}{W(\rho)}\left(W_{1,2}(x\,,\,\rho)W'_{2,1}(0\,,\,\rho)-W''_{1,2}(0\,,\,\rho)W_{2,1}(x\,,\,\rho)\right)\qquad(11)$$

where, $W(\rho)=-2\rho[1]$.

Derivative of solutions and asymptotic eigenvalues

Let us consider boundary value problem $L_1=L_1(\phi^2(x)\,,\,q(x)\,,\,b)$ for equation (1) with boundary conditions

$$C(0\,,\,\lambda)=1\,,\,C'(b,\lambda)=0\,,\,C'(0\,,\,\lambda)=0\qquad(12).$$

The boundary value problem L_1 for $b\in(x_1,x_2)$ has a countable set of positive eigenvalues.

Now for fixed $x\in(x_1\,,\,x_2)$ and use (7),(8) we determine the connection coefficients $T_1(\rho)\,,\,T_2(\rho)$

$$C(x\ ,\ \rho)=T_1(\rho)W_{1,1}+T_2(\rho)W_{2,1}\Rightarrow C'(x\ ,\ \rho)=T_1(\rho)W'_{1,1}+T_2(\rho)W'_{2,1}. \qquad (13)$$

$$For,.\quad x_1<x<x_2\Rightarrow\begin{cases}W_{1,1}(x\ ,\ \rho)=|\phi(x)|^{-\frac{1}{2}}\csc\pi\,\mu_1 e^{\rho\int_{x_1}^{x}|\phi(t)|dt}[1]\\ W_{2,1}(x\ ,\ \rho)=|\phi(x)|^{-\frac{1}{2}}\sin\pi\,\mu_1 e^{-\rho\int_{x_1}^{x}|\phi(t)|dt}[1]\end{cases} \qquad (14)$$

Let consider, $A=e^{\rho\int_{x_1}^{x}|\phi(t)|dt}[1]\Rightarrow W_{1,2}(x\ ,\ \rho)=|\phi(x)|^{-\frac{1}{2}}\csc\pi\,\mu_1 A$

The derivative of $W_{1,2}(x\ ,\ \rho)$ is following form

$$W'_{1,2}(x\ ,\ \rho)=\csc\pi\,\mu_1\left((|\phi(x)|^{-\frac{1}{2}})'A+|\phi(x)|^{-\frac{1}{2}}A'\right)$$

By use of fundamental theorem the derivative of A is as follow

$$\begin{cases}A'=\rho|\phi(x)|\left(e^{i\rho\int_{x_1}^{x}|\phi(t)|dt}[1]\right)=\rho|\phi(x)|B\\ B=e^{\rho\int_{x_1}^{x}|\phi(t)|dt-i\frac{\pi}{4}}[1]\end{cases} \qquad (15)$$

So $W'_{1,2}(x,\rho)$ is in following form

$$\begin{cases}W'_{1,2}(x,\rho)=\csc\pi\,\mu_1\left((|\phi(x)|^{-\frac{1}{2}})'A+\rho|\phi(x)|B\right)=\rho\csc\pi\,\mu_1\left(\frac{1}{\rho}(|\phi(x)|^{-\frac{1}{2}})'A+|\phi(x)|B\right),\\ C=(|\phi(x)|^{-\frac{1}{2}})',D=|\phi(x)|\Rightarrow W'_{1,2}(x,\rho)=\rho\csc\pi\,\mu_1(\frac{1}{\rho}CA+DB)\Rightarrow\\ W'_{1,2}(x,\rho)=\rho\csc\pi\,\mu_1 DB(1+\frac{1}{\rho}\frac{CA}{DB})\Rightarrow W'_{1,2}(x,\rho)=\rho\csc\pi\,\mu_1 DB\left(1+O(\frac{1}{\rho})\right)\end{cases} \qquad (16)$$

At last

$$W'_{1,2}(x,\rho)=\rho\csc\pi\,\mu_1 DB\left(1+O(\frac{1}{\rho})\right)$$

$$W'_{1,2}(x,\rho)=\rho\csc\pi\,\mu_1\,|\phi(x)|e^{-\rho\int_{x_1}^{0}|\phi(t)|dt}[1]\left(1+O(\frac{1}{\rho})\right) \qquad (17)$$

Similarly the derivative of $W_{2,1}(x,\rho)=|\phi(x)|^{-\frac{1}{2}}\sin\pi\,\mu_1 e^{-\rho\int_{x_1}^{x}|\phi(t)|dt}[1]$ is as following form so, Let us suppose

$$M = e^{-\rho\int_{x_2}^{x}|\phi(t)|dt}[1] \Rightarrow M' = -\rho|\phi(x)|e^{-\rho\int_{x1}^{0}|\phi(t)|dt}[1][1] = -\rho|\phi(x)|M \Rightarrow W_{2,1}(x,\rho) = |\phi(x)|^{-\frac{1}{2}}\sin\pi\mu_1 M$$

$$\begin{cases} W'_{2,1}(x,\rho) = \sin\pi\mu_1\left((|\phi(x)|^{-\frac{1}{2}})'M + |\phi(x)|^{-\frac{1}{2}}M' \right) = \\ \sin\pi\mu_1\left(KM - M|\phi(x)|^{\frac{1}{2}} \right) = \sin\pi\mu_1\left(\frac{1}{\rho}KM - ML \right) \\ K = (|\phi(x)|^{-\frac{1}{2}})', L = -|\phi(x)|^{\frac{1}{2}} \\ W'_{2,1}(x,\rho) = \rho\sin\pi\mu_1(ML)\left(1 - \frac{1}{\rho}(\frac{K}{L})\right) = \rho\sin\pi\mu_1(ML)\left(1 + O(\frac{1}{\rho})\right) \end{cases} \quad (18)$$

Finally $W'_{2,1}(x,\rho)$ is in form

$$W'_{2,1}(x,\rho) = -\rho\sin\pi\mu_1(ML)\left(1 - \frac{1}{\rho}(\frac{K}{L})\right) = -\rho\sin\pi\mu_1(e^{-\rho\int_{x1}^{0}|\phi(t)|dt}[1]|\phi(x)|^{\frac{1}{2}})\left(1 + O(\frac{1}{\rho})\right) \quad (19)$$

Hence we have estimated the solution of (1) defined by the initial condition (9) and Cramer's rule to determine the connection coefficients $T_1(\rho)$, $T_2(\rho)$ with

$$\begin{cases} C(x\ ,\ \rho) = T_1(\rho)W_{1,2}(x\ ,\ \rho) + T_2(\rho)W_{2,1}(x\ ,\ \rho) \Rightarrow 1 = T_1(\rho)W_{1,2}(0\ ,\ \rho) + T_2(\rho)W_{2,1}(0\ ,\ \rho) \\ C'(x\ ,\ \rho) = T_1(\rho)W'_{1,2}(x\ ,\ \rho) + T_2(\rho)VW'_{2,1}(x\ ,\ \rho) \Rightarrow 0 = T_1(\rho)W'_{1,2}(0\ ,\ \rho) + T_2(\rho)W'_{2,1}(0\ ,\ \rho) \end{cases} \quad (20)$$

$$\begin{cases} T_1(\rho) = \dfrac{W'_{2,1}(0\ ,\ \rho)}{-2\rho[1]} = -\sin\pi\mu_1|\phi(0)|^{\frac{1}{2}}(e^{-\rho\int_{x1}^{0}|\phi(t)|dt}[1])\left(1 + O(\frac{1}{\rho})\right) \\ T_2(\rho) = \dfrac{W'_{1,2}(0\ ,\ \rho)}{2\rho[1]} = \dfrac{1}{4}|\phi(0)|^{\frac{1}{2}}\csc\pi\mu_1\left(e^{\rho\int_{x_1}^{0}|\phi(t)|dt}[1]\right)\left(1 + O(\frac{1}{\rho})\right) \end{cases} \quad (21)$$

$$T_2(\rho) = \frac{1}{4}|\phi(0)|^{\frac{1}{2}}\csc\pi\mu_1\Gamma_1\ , T_1(\rho) = -\sin\pi\mu_1|\phi(0)|^{\frac{1}{2}}\Gamma_2 \quad (22)$$

$$\begin{cases} \Gamma_1 = \left(e^{\rho\int_{x1}^{0}|\phi(t)|dt}[1]\right)\ \Gamma_4 = (e^{\rho\int_{x_1}^{0}|\phi(t)|dt}[1])\left(1 + O(\frac{1}{\rho})\right) \\ \Gamma_3 = \left(e^{\rho\int_{x_1}^{0}|\phi(t)|dt}[1] - e^{-\rho\int_{x_1}^{0}|\phi(t)|dt}[1]\right)\left(1 + O(\frac{1}{\rho})\right),\ \Gamma_2 = (e^{-\rho\int_{x1}^{0}|\phi(t)|dt}[1]) \\ W'_{2,1}(x,\rho) = -\rho\sin\pi\mu_1|\phi(x)|^{\frac{1}{2}}\Gamma_4 \quad , W'_{1,2}(x,\rho) = \frac{1}{2}\rho\csc\pi\mu_1|\phi(x)|^{\frac{1}{2}}\Gamma_3 \end{cases} \quad (23)$$

By substituting (23)and (22) in (20) we obtain the leading term as $C'(x\ ,\ \rho)$ follows

$$\begin{cases} C'(x\ ,\ \rho) = -\frac{1}{4}|\phi(0)|^{\frac{1}{2}} \csc\pi\,\mu_1\Gamma_1 2i\rho\sin\pi\,\mu_1|\phi(x)|^{\frac{1}{2}}\Gamma_4 - \\ \qquad \frac{1}{2}\sin\pi\,\mu_1|\phi(0)|^{\frac{1}{2}}\Gamma_2 i\rho\csc\pi\,\mu_1|\phi(x)|^{\frac{1}{2}}\Gamma_3 = \\ \frac{1}{2}\sin\pi\,\mu_1|\phi(0)|^{\frac{1}{2}}\Gamma_2\rho\csc\pi\,\mu_1|\phi(x)|^{\frac{1}{2}}\Gamma_3 + \frac{1}{2}|\phi(0)|^{\frac{1}{2}}\csc\pi\,\mu_1\Gamma_1\rho\sin\pi\,\mu_1|\phi(x)|^{\frac{1}{2}}\Gamma_4 \\ C'(x\ ,\ \rho) = \frac{1}{2}|\phi(0)|^{\frac{1}{2}}\rho\sin\pi\,\mu_1\csc\pi\,\mu_1(\Gamma_2\Gamma_3 + \Gamma_1\Gamma_4). \end{cases} \quad (24)$$

Now must to determine the value of $\Gamma_2\Gamma_3 \, and \Gamma_1\Gamma_4$

$$\begin{cases} \Gamma_2\Gamma_3 = (e^{-\rho\int_{x_1}^{0}|\phi(t)|dt})\left(e^{\rho\int_{x_1}^{x}|\phi(t)|dt}[1] - e^{-\rho\int_{x_1}^{x}|\phi(t)|dt + i\frac{\pi}{4}}[1]\right)\left(1 + O(\frac{1}{\rho})\right), \\ \Gamma_1\Gamma_4 = (e^{-\rho\int_{x_1}^{x}|\phi(t)|dt}[1])\left(e^{\rho\int_{x_1}^{0}|\phi(t)|dt} - e^{-\rho\int_{x_1}^{0}|\phi(t)|dt + i\frac{\pi}{4}}\right)\left(1 + O(\frac{1}{\rho})\right) \end{cases} \quad (25)$$

By applying $C'(x\ ,\ \rho)$=0, consequently $\Gamma_2\Gamma_3 + \Gamma_1\Gamma_4$=0

$$\begin{cases} \Gamma_2\Gamma_3 = -\Gamma_1\Gamma_4 \Rightarrow (e^{-\rho\int_{x_1}^{0}|\phi(t)|dt})\left(e^{\rho\int_{x_1}^{x}|\phi(t)|dt}[1] - e^{-\rho\int_{x_1}^{x}|\phi(t)|dt + i\frac{\pi}{4}}[1]\right) = \\ \qquad (e^{-\rho\int_{x_1}^{x}|\phi(t)|dt}[1])\left(e^{-\rho\int_{x_1}^{0}|\phi(t)|dt} - e^{\rho\int_{x_1}^{0}|\phi(t)|dt}\right) \end{cases}$$

$$\begin{cases} [1](e^{\rho\int_{x_1}^{x}|\phi(t)|dt} - e^{-\rho\int_{x_1}^{x}|\phi(t)|dt})(e^{\rho\int_{x_1}^{x}|\phi(t)|dt}) = \\ (e^{\rho\int_{x_1}^{0}|\phi(t)|dt})(e^{-\rho\int_{x_1}^{0}|\phi(t)|dt} - e^{\rho\int_{x_1}^{0}|\phi(t)|dt}) \\ [1](e^{2\rho\int_{x_2}^{x}|\phi(t)|dt} - 1) = (1 - e^{2\rho\int_{x_1}^{0}|\phi(t)|dt}) \end{cases}$$

$$\begin{cases} [1] = \dfrac{1 - e^{2\rho\int_{x_1}^{0}|\phi(t)|dt}}{e^{2\rho\int_{x_1}^{x}|\phi(t)|dt} - 1} = 1 + O(\frac{1}{\rho}) \Rightarrow K = 1 - e^{2\rho\int_{x_1}^{0}|\phi(t)|dt} \\ \dfrac{K}{e^{2\rho\int_{x_1}^{x}|\phi(t)|dt} - 1} = 1 + O(\frac{1}{\rho}) \Rightarrow \dfrac{K}{1 + O(\frac{1}{\rho})} = e^{2\rho\int_{x_1}^{x}|\phi(t)|dt} - 1 \\ K + O(\frac{1}{\rho}) = e^{2\rho\int_{x_1}^{x}|\phi(t)|dt} - 1 \Rightarrow K + 1 + O(\frac{1}{\rho}) = e^{2\rho\int_{x_1}^{x}|\phi(t)|dt} \end{cases} \quad (26)$$

$$Ln\left(K+1+O(\frac{1}{\rho})\right)=O(\frac{1}{\rho^{\theta}})=2\rho\int_{x_1}^{x}\left|\phi(t)\right|dt-i\frac{\pi}{4}-2k\pi i \qquad (27)$$

By dividing (27) to $2\int_{x_1}^{x}\left|\phi(t)\right|dt$ I obtained the leading term ρ as follows

$$\rho_k=\frac{i\frac{\pi}{4}+2k\pi i}{2\int_{x_1}^{x}\left|\phi(t)\right|dt}+O(\frac{1}{\rho^{\theta}})=\frac{k\pi+\frac{\pi}{8}}{\int_{x_1}^{x}\left|\phi(t)\right|dt}+O(\frac{1}{\rho^{\theta}})$$

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